SOME HADAMARD-TYPE INEQUALITIES FOR COORDINATED $P-$CONVEX FUNCTIONS AND GODUNOVA-LEVIN FUNCTIONS

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Abstract. In this paper we established new Hadamard-type inequalities for functions that co-ordinated Godunova-Levin functions and co-ordinated $P-$convex functions, therefore we proved a new inequality involving product of convex functions and $P-$functions on the co-ordinates.

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and let $a, b \in I$, with $a < b$. The following inequality;

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

is known in the literature as Hadamard's inequality. Both inequalities hold in the reversed direction if f is concave.

In [1], E.K. Godunova and V.I. Levin introduced the following class of functions.

Definition 1. *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to belong to the class of $Q(I)$ if it is nonnegative and, for all $x, y \in I$ and $\lambda \in (0,1)$ satisfies the inequality;*

$$f(\lambda x + (1-\lambda)y) \leq \frac{f(x)}{\lambda} + \frac{f(y)}{1-\lambda}$$

In [2], S.S. Dragomir et.al., defined following new class of functions.

Definition 2. *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is P function or that f belongs to the class of $P(I)$, if it is nonnegative and for all $x, y \in I$ and $\lambda \in [0,1]$, satisfies the following inequality;*

$$f(\lambda x + (1-\lambda)y) \leq f(x) + f(y)$$

In [2], S.S. Dragomir et.al., proved two inequalities of Hadamard's type for class of Godunova-Levin functions and $P-$ functions.

Theorem 1. *Let $f \in Q(I), a, b \in I$, with $a < b$ and $f \in L_1[a,b]$. Then the following inequality holds.*

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$$f(\frac{a+b}{2}) \leq \frac{4}{b-a}\int_a^b f(x)dx \tag{1.1}$$

Theorem 2. *Let $f \in P(I), a, b \in I$, with $a < b$ and $f \in L_1[a,b]$. Then the following inequality holds.*

$$f(\frac{a+b}{2}) \leq \frac{2}{b-a}\int_a^b f(x)dx \leq 2[f(a)+f(b)] \tag{1.2}$$

In [10], Tunç proved following theorem which containing product of convex functions and $P-$functions.

Theorem 3. *Let $a, b \in [0, \infty)$, $a < b$, $I = [a, b]$ with $f, g : [a, b] \to \mathbb{R}$ be functions f, g and fg are in $L_1([a, b])$. If f is convex and g belongs to the class of $P(I)$ then,*

$$\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx \leq \frac{M(a,b)+N(a,b)}{2} \tag{1.3}$$

where $M(a,b) = f(a)\,g(a) + f(b)\,g(b)$ and $N(a,b) = f(a)\,g(b) + f(b)\,g(a)$.

In [3], S.S. Dragomir defined convexity on the co-ordinates, as following;

Definition 3. *Let us consider the bidimensional interval $\Delta = [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$, $c < d$. A function $f : \Delta \to \mathbb{R}$ will be called convex on the co-ordinates if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ are convex where defined for all $y \in [c,d]$ and $x \in [a,b]$. Recall that the mapping $f : \Delta \to \mathbb{R}$ is convex on Δ if the following inequality holds,*

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x,y) + (1-\lambda)f(z,w)$$

for all $(x,y), (z,w) \in \Delta$ and $\lambda \in [0,1]$.

Every convex function is co-ordinated convex but the converse is not generally true.

In [3], S.S. Dragomir established the following inequalities of Hadamard's type for co-ordinated convex functions on a rectangle from the plane $\mathbb{R}^2$.

Theorem 4. *Suppose that $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is convex on the co-ordinates on Δ. Then one has the inequalities;*

$$\begin{aligned}
& f(\frac{a+b}{2}, \frac{c+d}{2}) \\
\leq\ & \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x, \frac{c+d}{2})dx + \frac{1}{d-c}\int_c^d f(\frac{a+b}{2}, y)dy\right] \\
\leq\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy \\
\leq\ & \frac{1}{4}\left[\frac{1}{(b-a)}\int_a^b f(x,c)dx + \frac{1}{(b-a)}\int_a^b f(x,d)dx \right. \\
& \left. + \frac{1}{(d-c)}\int_c^d f(a,y)dy + \frac{1}{(d-c)}\int_c^d f(b,y)dy\right] \\
\leq\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}
\end{aligned} \tag{1.4}$$

For recent results which similar to above inequalities see [5], [6], [7], [8] and [9].

In [4], M.E. Ozdemir et.al., established the following Hadamard's type inequalities as above for co-ordinated $m-$convex and $(\alpha, m)-$convex functions.

Theorem 5. *Suppose that $f : \Delta = [0, b] \times [0, d] \to \mathbb{R}$ is $m-$convex on the co-ordinates on Δ. If $0 \le a < b < \infty$ and $0 \le c < d < \infty$ with $m \in (0, 1]$, then one has the inequality;*

$$\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dx dy \le \frac{1}{4(b-a)} \min\{v_1, v_2\} + \frac{1}{4(d-c)} \min\{v_3, v_4\} \tag{1.5}$$

where

$$\begin{aligned} v_1 &= \int_a^b f(x, c) dx + m \int_a^b f(x, \frac{d}{m}) dx \\ v_2 &= \int_a^b f(x, d) dx + m \int_a^b f(x, \frac{c}{m}) dx \\ v_3 &= \int_c^d f(a, y) dy + m \int_c^d f(\frac{b}{m}, y) dy \\ v_4 &= \int_c^d f(b, y) dy + m \int_c^d f(\frac{a}{m}, y) dy. \end{aligned}$$

Theorem 6. *Suppose that $f : \Delta = [0, b] \times [0, d] \to \mathbb{R}$ is $m-$convex on the co-ordinates on Δ. If $0 \le a < b < \infty$ and $0 \le c < d < \infty$, $m \in (0, 1]$ with $f_x \in L_1[0, d]$ and $f_y \in L_1[0, b]$, then one has the inequalities;*

$$\begin{aligned} &\frac{1}{b-a} \int_a^b f(x, \frac{c+d}{2}) dx + \frac{1}{d-c} \int_c^d f(\frac{a+b}{2}, y) dy \\ \le\ & \frac{1}{(b-a)(d-c)} \left[\int_a^b \int_c^d \frac{f(x, y) + m f(x, \frac{y}{m})}{2} dy dx \right. \\ & \left. + \int_c^d \int_a^b \frac{f(x, y) + m f(\frac{x}{m}, y)}{2} dx dy \right] \end{aligned} \tag{1.6}$$

Similar results can be found for $(\alpha, m)-$convex functions in [4]. In this paper we established new Hadamard-type inequalities for Godunova-Levin functions and $P-$functions on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$ and we proved a new inequality involving product of co-ordinated convex functions and co-ordinated $P-$functions.

2. MAIN RESULTS

We define Godunova-Levin functions and $P-$functions on the co-ordinates as the following:

Definition 4. *Let us consider the bidimensional interval $\Delta = [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$, $c < d$. A function $f : \Delta \to \mathbb{R}$ is said to belong to the class of $Q(I)$ if it is nonnegative and for all $(x, y), (z, w) \in \Delta$ and $\lambda \in (0, 1)$ satisfies the following*

inequality;

$$f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)\leq \frac{f(x,y)}{\lambda}+\frac{f(z,w)}{1-\lambda}$$

A function $f:\Delta\to\mathbb{R}$ is said to belong to the class of $Q(I)$ on Δ is called coordinated Godunova-Levin function if the partial mappings $f_y:[a,b]\to\mathbb{R}$, $f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}$, $f_x(v)=f(x,v)$ are belong to the class of $Q(I)$ where defined for all $y\in[c,d]$ and $x\in[a,b]$.

We denote this class of functions by $QX(f,\Delta)$. If the inequality reversed then f is said to be concave on Δ and we denote this class of functions by $QV(f,\Delta)$.

Definition 5. *Let $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ be a $P-$function with $a<b$, $c<d$. If it is nonnegative and for all $(x,y),(z,w)\in\Delta$ and $\lambda\in(0,1)$ the following inequality*

holds:

$$f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)\leq f(x,y)+f(z,w)$$

A function $f:\Delta\to\mathbb{R}$ is said to belong to the class of $P(I)$ on Δ is called coordinated $P-$function if the partial mappings $f_y:[a,b]\to\mathbb{R}$, $f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}$, $f_x(v)=f(x,v)$ are $P-$functions where defined for all $y\in[c,d]$ and $x\in[a,b]$.

We denote this class of functions by $PX(f,\Delta)$. We need following lemma for our main theorem.

Lemma 1. *Every f function that belongs to the class $Q(I)$ is said to belongs to class $QX(f,\Delta)$.*

Proof. Suppose that $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ is said to belong to the class $Q(I)$ on Δ. Consider the function $f_x:[c,d]\to[0,\infty)$, $f_x(v)=f(x,v)$. Then $\lambda\in(0,1)$ and $v_1,v_2\in[c,d]$, one has:

$$\begin{aligned}
f_x(\lambda v_1+(1-\lambda)v_2) &= f(x,\lambda v_1+(1-\lambda)v_2)\\
&= f(\lambda x+(1-\lambda)x,\lambda v_1+(1-\lambda)v_2)\\
&\leq \frac{f(x,v_1)}{\lambda}+\frac{f(x,v_2)}{1-\lambda}\\
&= \frac{f_x(v_1)}{\lambda}+\frac{f_x(v_2)}{1-\lambda}
\end{aligned}$$

which shows convexity of f_x. The fact that $f_y:[a,b]\to\mathbb{R}$, $f_y(u)=f(u,y)$ is also convex on $[a,b]$ for all $y\in[c,d]$ goes likewise and we shall omit the details. □

The following inequalities is considered the Hadamard-type inequalities for Godunova-Levin functions on the co-ordinates.

Theorem 7. *Suppose that $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ is said to belong to the class $QX(f,\Delta)$ on the co-ordinates on Δ with $f_x\in L_1[c,d]$ and $f_y\in L_1[a,b]$, then one has the inequalities:*

$$\begin{aligned}
&\frac{1}{16}\left[f(\frac{a+b}{2},\frac{c+d}{2})\right] \qquad (2.1)\\
&\leq \frac{1}{8}\left[\frac{1}{b-a}\int_a^b f(x,\frac{c+d}{2})dx+\frac{1}{d-c}\int_c^d f(\frac{a+b}{2},y)dy\right]\\
&\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx
\end{aligned}$$

Proof. Since $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is said to belong to the class $QX(f,\Delta)$ on the co-ordinates it follows that the mapping $g_x : [c,d] \to \mathbb{R}$, $g_x(y) = f(x,y)$ is Godunova-Levin function on $[c,d]$ for all $x \in [a,b]$. Then by Hadamard's inequality (1.1) one has:

$$g_x(\frac{c+d}{2}) \leq \frac{4}{d-c}\int_c^d g_x(y)dy, \forall x \in [a,b].$$

That is,

$$f(x,\frac{c+d}{2}) \leq \frac{4}{d-c}\int_c^d f(x,y)dy, \forall x \in [a,b].$$

Integrating this inequality on $[a,b]$, we have:

$$\frac{1}{b-a}\int_a^b f(x,\frac{c+d}{2})dx \leq \frac{4}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \tag{2.2}$$

A similar argument applied for the mapping $g_y : [a,b] \to \mathbb{R}$, $g_y(x) = f(x,y)$, we get:

$$\frac{1}{d-c}\int_c^d f(\frac{a+b}{2},y)dy \leq \frac{4}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)dxdy \tag{2.3}$$

Summing the inequalities (2.2), and (2.3) , we get the last inequality in (2.1).

Therefore, by Hadamard's inequality (1.1) we also have:

$$f(\frac{a+b}{2},\frac{c+d}{2}) \leq \frac{4}{d-c}\int_c^d f(\frac{a+b}{2},y)dy$$

and

$$f(\frac{a+b}{2},\frac{c+d}{2}) \leq \frac{4}{b-a}\int_a^b f(x,\frac{c+d}{2})dx$$

which give, by addition the first inequality in (2.1).

This completes the proof. $\square$

Corollary 1. *Suppose that $f : \Delta = [a,b] \times [a,b] \to \mathbb{R}$ is said to belong to the class $QX(f,\Delta)$ on the co-ordinates, then one has the inequalities:*

$$\begin{aligned} \frac{1}{16}\left[f(\frac{a+b}{2},\frac{a+b}{2})\right] &\leq \frac{1}{8}\left[\frac{1}{b-a}\int_a^b\left\{f(x,\frac{a+b}{2})+f(\frac{a+b}{2},x)\right\}dx\right] \\ &\leq \frac{1}{(b-a)^2}\int_a^b\int_a^b f(x,y)dydx \end{aligned} \tag{2.4}$$

Corollary 2. *In (2.1), under the assumptions Theorem 4 with $f(x,y) = f(y,x)$ for all $x \in [a,b] \times [a,b]$, we have:*

$$\begin{aligned} f(\frac{a+b}{2},\frac{a+b}{2}) &\leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,\frac{a+b}{2})dx\right] \\ &\leq \frac{1}{(b-a)^2}\int_a^b\int_a^b f(x,y)dydx \end{aligned}$$

Lemma 2. *Every $P-$functions are coordinated on Δ or belong to the class of $PX(f,\Delta)$.*

Proof. Let f be a $P-$function and defined by $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ where $y \in [c,d]$, $x \in [a,b]$ and $\lambda \in [0,1], v_1, v_2 \in [a,b]$, then

$$\begin{aligned}
f_x(\lambda v_1 + (1-\lambda)v_2) &= f(x, \lambda v_1 + (1-\lambda)v_2) \\
&= f(\lambda x + (1-\lambda)x, \lambda v_1 + (1-\lambda)v_2) \\
&\leq f(x, v_1) + f(x, v_2) \\
&= f_x(v_1) + f_x(v_2)
\end{aligned}$$

which shows convexity of f_x. The fact that $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ is also convex on $[a,b]$ for all $y \in [c,d]$ goes likewise and we shall omit the details. □

The following inequalities is considered the Hadamard-type inequalities for P-functions on the co-ordinates.

Theorem 8. *Suppose that $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is said to belong to the class $PX(f,\Delta)$ on the co-ordinates on Δ with $f_x \in L_1[c,d]$ and $f_y \in L_1[a,b]$, then one has the inequalities:*

$$\begin{aligned}
(2.5)\quad f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy \\
&\leq \frac{4}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \\
&\leq \frac{2}{(b-a)}\left[\int_a^b f(x,c)dx + \int_a^b f(x,d)dx\right] \\
&\quad + \frac{2}{(d-c)}\left[\int_c^d f(a,y)dy + \int_c^d f(b,y)dy\right]
\end{aligned}$$

Proof. Since $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is said to belong to the class $PX(f,\Delta)$ on the co-ordinates it follows that the mapping $g_x : [c,d] \to \mathbb{R}$, $g_x(y) = f(x,y)$ is $P-$function on $[c,d]$ for all $x \in [a,b]$. Then by Hadamard's inequality (1.2) one has:

$$f\left(x, \frac{c+d}{2}\right) \leq \frac{2}{d-c}\int_c^d f(x,y)dy \leq 2\left[f(x,c) + f(x,d)\right]$$

Integrating this inequality on $[a,b]$, we have:

$$\begin{aligned}
(2.6)\quad \frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx &\leq \frac{2}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \\
&\leq \frac{2}{b-a}\left[\int_a^b f(x,c)dx + \int_a^b f(x,d)dx\right]
\end{aligned}$$

A similar argument applied for the mapping $g_y : [a,b] \to \mathbb{R}$, $g_y(x) = f(x,y)$, we get:

$$\begin{aligned}
(2.7)\quad \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy &\leq \frac{2}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)dxdy \\
&\leq \frac{2}{d-c}\left[\int_c^d f(a,y)dy + \int_c^d f(b,y)dy\right]
\end{aligned}$$

Addition (2.6) and (2.7), we get:

$$\begin{aligned}\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx &\leq \frac{1}{2(b-a)}\left[\int_a^b f(x,c)dx+\int_a^b f(x,d)dx\right]\\ &+\frac{1}{2(d-c)}\left[\int_c^d f(a,y)dy+\int_c^d f(b,y)dy\right]\end{aligned}$$

Which gives the last inequality in (2.5). We also have:

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(x,\frac{c+d}{2})dx+\frac{1}{d-c}\int_c^d f(\frac{a+b}{2},y)dy\\ &\leq \frac{4}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)dxdy\end{aligned}$$

Which gives the mid inequality in (2.5). By Hadamard's inequality we also have:

$$f(\frac{a+b}{2},\frac{c+d}{2})\leq \frac{2}{b-a}\int_a^b f(x,\frac{c+d}{2})dx$$

and

$$f(\frac{a+b}{2},\frac{c+d}{2})\leq \frac{2}{d-c}\int_c^d f(\frac{a+b}{2},y)dy$$

Adding these inequalities we get,

$$f(\frac{a+b}{2},\frac{c+d}{2})\leq \frac{1}{b-a}\int_a^b f(x,\frac{c+d}{2})dx+\frac{1}{d-c}\int_c^d f(\frac{a+b}{2},y)dy$$

Which gives the first inequality in (2.5). This completes the proof. □

Theorem 9. *Let* $a,b,c,d\in[0,\infty)$,$a<b$ *and* $c<d$, $\Delta=[a,b]\times[c,d]$ *with* $f,g:\Delta\to\mathbb{R}$ *be functions* f,g *and* fg *are in* $L_1([a,b]\times[c,d])$. *If* f *is co-ordinated convex and* g *belongs to the class of* $PX(f,\Delta)$, *then one has the inequality;*

$$\begin{aligned}&\frac{1}{(d-c)(b-a)}\int_a^b\int_c^d f(x,y)\,g(x,y)\,dydx\\ \leq\;&\frac{L(a,b,c,d)+M(a,b,c,d)+N(a,b,c,d)}{4}\end{aligned} \tag{2.9}$$

where

$$\begin{aligned}L(a,b,c,d)&=f(a,c)g(a,c)+f(b,c)g(b,c)+f(a,d)g(a,d)+f(b,d)g(b,d)\\ M(a,b,c,d)&=f(a,c)g(a,d)+f(a,d)g(a,c)+f(b,c)g(b,d)+f(b,d)g(b,c)\\ &\quad+f(b,c)g(a,c)+f(b,d)g(a,d)+f(a,c)g(b,c)+f(a,d)g(b,d)\\ N(a,b,c,d)&=f(b,c)g(a,d)+f(b,d)g(a,c)+f(a,c)g(b,d)+f(a,d)g(b,c)\end{aligned}$$

Proof. Since f is co-ordinated convex and g belongs to the class of $PX(f,\Delta)$, by using partial mappings and from inequality (1.3), we can write

$$\frac{1}{d-c}\int_c^d f_x(y)\,g_x(y)\,dy\leq\frac{f_x(c)\,g_x(c)+f_x(d)\,g_x(d)+f_x(c)\,g_x(d)+f_x(d)\,g_x(c)}{2}$$

That is

$$\frac{1}{d-c}\int_c^d f(x,y)\,g(x,y)\,dy\leq\frac{f(x,c)\,g(x,c)+f(x,d)\,g(x,d)+f(x,c)\,g(x,d)+f(x,d)\,g(x,c)}{2}$$

Dividing both sides of this inequality $(b-a)$ and integrating over $[a,b]$ respect to x, we have

$$\begin{aligned} & \frac{1}{(d-c)(b-a)}\int_a^b\int_c^d f(x,y)\,g(x,y)\,dydx \\ \leq\;& \frac{1}{2(b-a)}\int_a^b f(x,c)\,g(x,c)+\frac{1}{2(b-a)}\int_a^b f(x,d)\,g(x,d) \\ & +\frac{1}{2(b-a)}\int_a^b f(x,c)\,g(x,d)+\frac{1}{2(b-a)}\int_a^b f(x,d)\,g(x,c) \end{aligned} \tag{2.10}$$

By applying (1.3) to each integral on right hand side of (2.10) and using these inequalities in (2.10), we get the required result as following

$$\begin{aligned} & \frac{1}{(d-c)(b-a)}\int_a^b\int_c^d f(x,y)\,g(x,y)\,dydx \\ \leq\;& \frac{f(a,c)\,g(a,c)+f(b,c)\,g(b,c)+f(a,c)\,g(b,c)+f(b,c)\,g(a,c)}{4} \\ & +\frac{f(a,d)\,g(a,d)+f(b,d)\,g(b,d)+f(a,d)\,g(b,d)+f(b,d)\,g(a,d)}{4} \\ & \frac{f(a,c)\,g(a,d)+f(b,c)\,g(b,d)+f(a,c)\,g(b,d)+f(b,c)\,g(a,d)}{4} \\ & +\frac{f(a,d)\,g(a,c)+f(b,d)\,g(b,c)+f(a,d)\,g(b,c)+f(b,d)\,g(a,c)}{4} \end{aligned}$$

By a similar argument, if we apply (1.3) for $f_y(x)g_y(x)$ on $[a,b]$, we get the same result. □

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